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The Properties of the Lucas Triangle and its Relationship to other Integer Sequences

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ABSTRACT

This study focuses on the construction and analysis of the Lucas Triangle as a recursive triangular array of integers related to Pascal's Triangle and Lucas numbers. The study establishes the formation of the Lucas Triangle and investigates its mathematical properties through the development and proof of various theorems, propositions, and lemmas. Several combinatorial and divisibility properties of the triangle are derived, including closed-form representations and row sum identities. Furthermore, the study explores how the entries of the Lucas Triangle generate significant integer sequences such as natural numbers, odd numbers, square numbers, Lucas numbers, Fibonacci numbers, central binomial coefficients, triangular numbers, Catalan numbers, and square pyramidal numbers. To support computation and visualization, Python programming in Google Colab was utilized to generate and analyze the entries of the Lucas Triangle. The findings reveal that the Lucas Triangle possesses rich recursive and combinatorial structures that establish meaningful relationships among various integer sequences. The study contributes to the broader investigation of triangular arrays and recursive number patterns in discrete mathematics and number theory.

Keywords: *Triangular Array, Pascal Triangle, Recursion, Combinatorics, Lucas Number*

INTRODUCTION

This study focused on the construction and analysis of the Lucas Triangle and its relationship to other integer sequences. The Lucas Triangle is a Pascal-like triangular array associated with the Lucas sequence and is formed through a recursive structure. Since Pascal's Triangle has important applications in number theory, combinatorics, and binomial expansion, the Lucas Triangle offers another mathematical structure that may reveal significant patterns among integer sequences.

The study recognized the importance of examining the entries of the Lucas Triangle to identify their properties and relationships with well-known integer sequences. Specifically, it explored how the triangle generates or represents natural numbers, odd numbers, square numbers, Lucas numbers, Fibonacci numbers, central binomial coefficients, triangular numbers, Catalan numbers, and square pyramidal numbers. Through this investigation, the study aimed to provide a systematic treatment of the Lucas Triangle as a mathematical structure in number theory and combinatorics.

METHODS

The study employed a descriptive-expository research design. The descriptive component was used to define the Lucas Triangle, generate its initial rows, observe numerical patterns, and examine relationships among

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its entries. The expository component was used to present mathematical arguments and prove the identified properties through direct proof, mathematical induction, recurrence relations, generating functions, and elementary number-theoretic reasoning.

Python programming using Google Colab was also utilized to support the computation and visualization of the Lucas Triangle. The program was used to generate entries, verify derived results, and illustrate numerical patterns. However, the programming component was limited to numerical generation and visualization only.

RESULTS AND DISCUSSION

Construction of the Lucas Triangle

The study established the Lucas Triangle as a Pascal-type triangular array generated recursively by $\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m \end{bmatrix} + \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}$, subject to the boundary conditions $\begin{bmatrix} n \\ 0 \end{bmatrix} = 1$, $\begin{bmatrix} n \\ n \end{bmatrix} = 2$, and $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 2$.
Using these conditions, the entries of the Lucas Triangle were systematically generated row by row. The recursive construction demonstrated that every interior entry is obtained as the sum of the two adjacent entries from the preceding row. This construction provided the foundation for deriving the various properties and sequence relationships investigated in the study.

Properties of the Lucas Triangle

Several properties of the Lucas Triangle were established through the formulation and proof of theorems, propositions, and lemmas.

1. A closed-form formula for the entries of the Lucas Triangle was obtained: $\begin{bmatrix} n \\ m \end{bmatrix} = \binom{n}{m} + \binom{n-1}{m-1}$where C(n,m) denotes the binomial coefficient.
2. An equivalent representation was also established: $\begin{bmatrix} n \\ m \end{bmatrix} = \binom{n}{m} \cdot \frac{(n+m)}{n}$
3. The study further showed that the sum of all entries in the nth row satisfies $\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} = 3(2^{n-1})$
4. Likewise, the alternating sum of the entries in each row was shown to satisfy $\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} = 0$.
5. Additional parity and divisibility properties were derived, revealing the arithmetic and combinatorial structure of the Lucas Triangle and its close relationship with Pascal's Triangle.

Relationship Between the Lucas Triangle to other Integer Sequences

1. (Natural Numbers) The study established that the first column of the Lucas Triangle generates the natural numbers. Specifically, $\begin{bmatrix} n \\ 1 \end{bmatrix} = n + 1$. This result shows that the entries in the first column form the sequence $2, 3, 4, 5, 6, 7, \ldots$ which corresponds to the consecutive natural numbers greater than one.
2. (Odd Numbers) It was shown that the entries in the penultimate column generate the odd numbers according to the formula $\begin{bmatrix} n \\ n-1 \end{bmatrix} = 2n - 1$. Consequently, the sequence $1, 3, 5, 7, 9, 11, \ldots$ appears naturally within the structure of the Lucas Triangle.
3. (Square Numbers) The study established that the entries in the third diagonal from the right generate square numbers through $\begin{bmatrix} n \\ n-2 \end{bmatrix} = (n-1)^2$. This result produces the sequence $1, 4, 9, 16, 25, 36, \ldots$ which corresponds to the perfect squares.
4. (Lucas Numbers) The Lucas numbers were shown to arise from diagonal summations of the Lucas Triangle. Specifically, $L_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \begin{bmatrix} n-i \\ i \end{bmatrix}$. This identity demonstrates a direct connection between the Lucas Triangle and the Lucas sequence.

5. The study established two identities involving Fibonacci numbers.
   a. The even-indexed Fibonacci numbers satisfy $F_{2n} = \sum_{i=0}^{n} \begin{bmatrix} n+i \\ 2i \end{bmatrix}$.
   b. Similarly, the odd-indexed Fibonacci numbers satisfy $F_{2n+1} = \sum_{i=0}^{n-1} \begin{bmatrix} n+i \\ 2i+1 \end{bmatrix}$.

   These identities reveal that Fibonacci numbers may be generated through specific diagonal summations of the Lucas Triangle.
6. (Central Binomial Coefficients) The study showed that the central entries of the Lucas Triangle are related to the central binomial coefficients through $\begin{bmatrix} 2n \\ n \end{bmatrix} = 3\binom{2n-1}{n}$. This identity establishes a direct combinatorial relationship between the Lucas Triangle and the central binomial coefficients.
7. (Triangular Numbers) The triangular numbers were generated through the sum of the first three entries in the nth row according to $T_{n+1} = \begin{bmatrix} n \\ 0 \end{bmatrix} + \begin{bmatrix} n \\ 1 \end{bmatrix} + \begin{bmatrix} n \\ 2 \end{bmatrix}$. This result demonstrates that triangular numbers are naturally embedded within the row structure of the Lucas Triangle.
8. (Catalan Number) The study established that Catalan numbers may be obtained from the difference of two adjacent central entries: $C_n = \begin{bmatrix} 2n \\ n \end{bmatrix} - \begin{bmatrix} 2n \\ n+1 \end{bmatrix}$. This identity provides another combinatorial interpretation of Catalan numbers through the Lucas Triangle.
9. (Square Pyramidal Numbers) The square pyramidal numbers were shown to satisfy $SP_n = \begin{bmatrix} n+2 \\ n-1 \end{bmatrix}$. Thus, the sequence $1, 5, 14, 30, 55, \ldots$ appears naturally among the entries of the Lucas Triangle.
10. Computational Verification Using Python. To verify the theoretical results, Python programming using Google Colab was employed to generate the entries of the Lucas Triangle and evaluate the derived formulas. The computational outputs were consistent with all established theorems and identities. Moreover, the generated visualizations provided additional evidence of the structural patterns and sequence relationships embedded within the Lucas Triangle.

Overall, the findings demonstrate that the Lucas Triangle possesses a rich recursive, combinatorial, and arithmetic structure and serves as a unified framework for generating numerous classical integer sequences.

CONCLUSION

The study concluded that the Lucas Triangle is a meaningful recursive mathematical structure closely related to Pascal's Triangle and the Lucas sequence. Its entries possess important combinatorial, recurrence-based, and divisibility properties.

The study further concluded that the Lucas Triangle can serve as a unified framework for generating and representing several integer sequences, including natural numbers, odd numbers, square numbers, Lucas numbers, Fibonacci numbers, central binomial coefficients, triangular numbers, Catalan numbers, and square pyramidal numbers.

Overall, the Lucas Triangle provides a useful structure for exploring relationships among integer sequences and contributes to the study of triangular arrays, number patterns, and recursive structures in number theory and combinatorics.

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