ISSN: 3116-3475
International Journal
of Education, Research, and Innovation Perspectives
Volume 2 Issue 7 (July 2026)

On Octagonal Pyramidal Number

Armando, D, Basilan
Eulogio "Amang" Rodriguez Institute of Science and Technology - Manila
armando.basilan@deped.gov.ph

Date Submitted:
February 27, 2026

Date Accepted:
April 14, 2026

Date Published:
July 13, 2026

DOI:
10.5281/zenodo.21338008

ABSTRACT

This study investigates octagonal pyramidal numbers, a three-dimensional figurate sequence obtained by summing successive octagonal numbers. Using descriptive and expository methods, it establishes the general formula $P_8(n) = \frac{n(n+1)(2n-1)}{2}$ and examines its equivalent algebraic and combinatorial forms. The study also derives ordinary and exponential generating functions, recurrence relations, summation identities, reciprocal sums, divisibility patterns, and parity properties. Furthermore, relationships between octagonal pyramidal numbers and other integer sequences, including triangular, cubic, odd, pentagonal, pentagonal pyramidal, tetrahedral, and heptagonal numbers, are presented and proven. The findings demonstrate that octagonal pyramidal numbers possess significant algebraic, combinatorial, and number-theoretic structures. These results provide a systematic foundation for further investigations involving generalized figurate numbers, computational properties, and possible applications in mathematics.

Keywords: *Octagonal Numbers, Figurate Numbers, Generating Function, Recurrence Relations, Number Theory*

INTRODUCTION

This study focuses on the construction and analysis of octagonal pyramidal numbers and their relationships with other integer sequences. Figurate numbers provide a geometric representation of numerical patterns by arranging equally spaced points into regular polygons and polyhedra. Polygonal numbers represent two-dimensional configurations, while pyramidal numbers extend these arrangements into three dimensions by successively stacking polygonal layers. An octagonal pyramidal number is therefore obtained by adding consecutive octagonal numbers, producing the sequence $1,9,30,70,135,231,364,540, \ldots$.

The study of figurate numbers is important in number theory and combinatorics because it connects geometric arrangements with polynomial formulas, finite sums, recurrence relations, generating functions, binomial coefficients, and Diophantine equations. Classical sequences such as triangular, square, pentagonal, tetrahedral, and square pyramidal numbers have been widely examined and have generated numerous identities and combinatorial interpretations. General treatments of polygonal and pyramidal numbers have also provided formulas for broad families of figurate sequences (Deza & Deza, 2012; Sloane & Plouffe, 1995).

Despite these developments, studies devoted specifically to octagonal pyramidal numbers remain comparatively limited. Existing works have generally discussed pyramidal numbers as members of generalized figurate families, investigated equal values between different polygonal and pyramidal sequences, or examined selected Diophantine properties (Brindza et al., 1998; Kovács & Rábai, 2018). However, the available literature provides relatively little integrated discussion of the closed forms, combinatorial representations, generating functions, reciprocal sums, recurrence relations, divisibility behavior, parity patterns, and sequence identities of octagonal pyramidal numbers within a single mathematical exposition.

ISSN: 3116-3475
International Journal
of Education, Research, and Innovation Perspectives
Volume 2 Issue 7 (July 2026)

Addressing this gap is significant because octagonal pyramidal numbers provide a useful structure for investigating how a cubic integer sequence may be represented in several algebraically equivalent forms. Their properties illustrate how geometric summation can produce polynomial identities, while their generating functions and recurrence relations demonstrate important techniques in enumerative combinatorics. Their relationships with triangular, cubic, pentagonal, and tetrahedral sequences also reveal that figurate numbers are not isolated constructions but members of a broader interconnected system of integer sequences.

Accordingly, this study aims to provide a systematic treatment of octagonal pyramidal numbers. Specifically, it establishes their general formula, derives selected algebraic and combinatorial properties, determines their generating functions and recurrence relations, examines divisibility and parity behavior, and identifies their relationships with other well-known integer sequences. Through this investigation, the study contributes to the continuing examination of figurate numbers in number theory and combinatorics.

METHODS

The study employed descriptive and expository methods of mathematical research. The descriptive method was used to define and present the fundamental concepts, structures, numerical patterns, and geometric interpretations associated with octagonal pyramidal numbers. The expository method was used to formulate and establish mathematical properties through definitions, theorems, observations, examples, and proofs. The proofs employed algebraic manipulation, direct proof, mathematical induction, summation identities, binomial coefficients, generating-function techniques, partial-fraction decomposition, and modular arithmetic. Previously established concepts involving figurate numbers, pyramidal numbers, recurrence relations, and generating functions were used as the theoretical foundation of the investigation.

RESULTS AND DISCUSSION

Construction and General Formula of Octagonal Pyramidal Numbers

The n^{th} octagonal number is defined by $O_n = n(3n-2)$.

The nth octagonal pyramidal number, denoted by $P_8(n)$, is obtained by summing the first n octagonal numbers: $P_8(n) = \sum_{k=1}^{n} O_k = \sum_{k=1}^{n} k(3k-2)$

Using the standard formulas $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ and $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ it follows that $\sum_{k=1}^{n} k(3k-2) = \frac{n(n+1)(2n-1)}{2}$. Thus, the general term of the octagonal pyramidal sequence is $P_8(n) = \frac{n(n+1)(2n-1)}{2}$

For $n = 1,2,3,4,$ and 5, the formula produces $P_8(1) = 1, P_8(2) = 9, P_8(3) = 30, P_8(4) = 70,$ and $P_8(5) = 135$. These values agree with the successive partial sums of the octagonal numbers $1,8,21,40,$ and 65.

Properties of Octagonal Pyramidal Numbers

Several properties of Octagonal Pyramidal numbers arise such as closed forms, recursive relations, and combinatorial notations understand the behavior of Octagonal Pyramidal numbers. The following properties are as follows.

- For any positive integer n, the nth octagonal pyramidal number is given by

$$P_8(n) = (2n-1)\binom{n+1}{2}.$$

- For any positive $n \geq 2$,

$$P_8(n) = \binom{n+2}{3} + 5\binom{n+1}{3}.$$

- The generating function of the Octagonal Pyramidal number can be written as

$$G(x) = \frac{x(1+5x)}{(1-x)^4}$$

- The exponential generating function of the Octagonal Pyramidal number is given by

$$G(x) = \frac{x(2+7x+2x^2)e^x}{2}$$

- For any positive integer n,

$$\sum_{n=1}^{\infty} \frac{1}{P_8(n)} = \frac{2(4\log(2)-1)}{3}$$

- For any positive integer n,

$$P_8(n) = \sum_{i=0}^{n-1} (n-i)(6i+1)$$

- For any positive integer n,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{P_8(n)} = \frac{2(\pi+1-4\log(2))}{3}.$$

- For every positive integer n, the nth octagonal pyramidal number is divisible by 3 except the $(3k-2)^{th}$ term. In symbols,

$$P_8(n) \equiv 0(mod\ 3)$$

whenever $n \neq 3k-2$ for any positive integer k.

- For every positive integer k,

$$P_8(n) = \begin{cases} P_8(n) \equiv 0\ mod\ 2, & n = 4k\ or\ 4k-1 \\ P_8(n) \not\equiv 0\ mod\ 2, & n = 4k-2\ or\ 4k-3 \end{cases}$$

- For any positive integer $n > 1$, the nth Octagonal Pyramidal number can be written as the sum of $(n-1)$th Octagonal Pyramidal Number and $n(3n-2)$. In symbol,

$$P_8(n) = P_8(n-1) + n(3n-2).$$

Where $P_8(1) = 1$.

- For any positive integer $n > 3$,

$$P_8(n) = 3P_8(n-1) - 3P_8(n-2) + P_8(n-3) + 6.$$

where $P_8(1) = 1$, $P_8(2) = 9$,and $P_8(3) = 30$.

- For any positive integer $n > 4$,

$$P_8(n) = 4P_8(n-1) - 6P_8(n-2) + 4P_8(n-3) - P_8(n-4).$$

where $P_8(1) = 1$, $P_8(2) = 9$, $P_8(3) = 30$, and $P_8(4) = 70$.

Relationship of Octagonal Pyramidal number to other integer sequences

Relationship of Octagonal Pyramidal number to other integer sequences such as triangular, cubic, pentagonal, pentagonal pyramidal, tetrahedral, and heptagonal number were also established.

- For any positive integer $n > 1$, the nth Octagonal Pyramidal number can be written as the sum of nth cubic number and $(n-1)$th triangular number. In symbols,

$$P_8(n) = n^3 + T_{n-1}$$

Where $T_1 = 1$.

- For any positive integer n, the nth negative indexed Octagonal Pyramidal number is the difference between the nth triangular number and nth cubic number. In symbols,

$$P_8(-n) = T_n - n^3$$

- For any positive integer n, the nth octagonal pyramidal number can be written as the product of nth odd number and nth triangular numbers. In symbols,

$$P_8(n) = O_n \cdot T_n.$$

- For any positive integer $n > 1$,

$$P_8(n) = nP_n - P_5(n-1)$$

Where P_n is the nth pentagonal number, $P_5(n)$ is the nth pentagonal pyramidal number and $P_5(1) = 1$.

- For any positive integer $n \geq 2$,

$$P_8(n) = Te_n + 5Te_{n-1}$$

Where Te_n is the nth Tetrahedral number and $Te_1 = 1$.

- For any positive integer $n > 1$, then

$$P_8(n) = H_n + Te_{n-1}$$

Where H_n is the nth Heptagonal pyramidal number and Te_n is the nth Tetrahedral pyramidal number and $Te_1 = 1$.

- For any positive integer $n > 1$,

$$P_8(n) = T_n + 6Te_{n-1}$$

whereT_n is the nth triangular number, and Te_n is the nth Tetrahedral number and $Te_1 = 1$.

CONCLUSION

The study concludes that octagonal pyramidal numbers form a significant three-dimensional figurate sequence generated by the successive summation of octagonal numbers. Their general term is given by $P_8(n) = \frac{n(n+1)(2n-1)}{2}$which provides the basis for their algebraic, combinatorial, and number-theoretic properties.

The study further concludes that octagonal pyramidal numbers possess ordinary and exponential generating functions, finite summation representations, reciprocal-sum identities, recurrence relations, divisibility conditions, and predictable parity patterns. These properties demonstrate that the sequence may be studied using several important techniques in number theory and combinatorics.

Moreover, octagonal pyramidal numbers are closely related to cubic, triangular, odd, pentagonal, pentagonal pyramidal, tetrahedral, and heptagonal pyramidal numbers. These identities demonstrate that octagonal pyramidal numbers are not isolated objects but form part of a broader network of interconnected figurate and integer sequences.

Overall, the findings provide a systematic mathematical foundation for further investigations involving generalized polygonal pyramidal numbers, combinatorial interpretations, congruence properties, sequence intersections, and higher-dimensional extensions.

References

Brindza, B., Pintér, Á., & Turjányi, S. (1998). On equal values of pyramidal and polygonal numbers. Indagationes Mathematicae, 9(2), 183–185.

Deza, E., & Deza, M. M. (2012). Figurate numbers. World Scientific.

Kovács, T., & Rábai, Z. (2018). Equal values of pyramidal numbers. Indagationes Mathematicae, 29(4), 1157–1166.

Muktibodh, A., Sheikh, U., Juneja, D., Barhate, R., & Miglani, Y. (2007). Sequences of pyramidal numbers. Scientia Magna, 3(1), 39–45.

OEIS Foundation. (n.d.). A002414: Octagonal pyramidal numbers. The On-Line Encyclopedia of Integer Sequences.

Sloane, N. J. A., & Plouffe, S. (1995). The encyclopedia of integer sequences. Academic Press