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On Singular Value Decomposition

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ABSTRACT

This study investigates the algebraic and geometric structures underlying Singular Value Decomposition (SVD), one of the most fundamental matrix factorizations in linear algebra with extensive applications in both theoretical and applied mathematics. The research aims to define SVD and derive its formal construction, identify the algebraic properties and conditions that guarantee its existence and uniqueness, provide a geometric interpretation of SVD through orthogonal transformations and ellipsoids, and demonstrate how its mathematical structure contributes to solving practical problems, particularly in data analysis. Employing a theoretical research approach, the study establishes the necessary foundations in linear algebra, including orthogonality, eigenvalues, and the Spectral Theorem, and presents a constructive proof of the existence of SVD for any real matrix by analyzing the symmetric positive semi-definite matrix ($A^T A$). The study further illustrates the geometric interpretation of SVD as a sequence of rotation, scaling, and rotation transformations that map the unit circle or sphere into an ellipsoid. To demonstrate its practical relevance, a computational example involving student grade data is analyzed using SVD. The findings show that the existence of SVD is guaranteed for every real matrix, with singular values being unique when arranged in descending order, while singular vectors may not be unique in the presence of repeated singular values. Geometrically, SVD is shown to represent linear transformations through orthogonal rotations and principal-axis scaling, providing a clear visualization of how matrices transform space. Moreover, the decomposition effectively identifies dominant and secondary patterns in student performance data by separating the data into left singular vectors, singular values, and right singular vectors, thereby revealing underlying latent structures. The study concludes that SVD serves as a powerful bridge between algebraic theory and geometric intuition, explaining concepts such as rank and dimensionality through the mapping of a unit sphere to an ellipsoid. Furthermore, its rank-one matrix decomposition forms the theoretical basis for low-rank approximation, making SVD an indispensable mathematical tool for pattern extraction, dimensionality reduction, noise filtering, and modern computational analysis.

Keywords: *Singular Value Decomposition (SVD), Linear Algebra, Matrix Factorization, Spectral Theorem, Orthogonal Transformations, Singular Values, Singular Vectors, Geometric Interpretation, Low-Rank Approximation, Dimensionality Reduction, Data Analysis.*

INTRODUCTION

Singular Value Decomposition (SVD) is widely recognized as one of the most significant matrix factorizations in modern linear algebra because of its broad theoretical importance and extensive practical applications. Unlike many classical matrix decompositions that require matrices to be square or symmetric, SVD exists for every real or complex matrix regardless of its dimensions. This universality has established SVD as an indispensable mathematical tool in numerical analysis, optimization, machine learning, statistics, signal processing, image compression, and many other scientific disciplines. Beyond its computational efficiency, SVD provides a comprehensive description of the intrinsic structure of a matrix by expressing every linear transformation as the product of two orthogonal transformations and a diagonal scaling matrix. Consequently, it serves as a bridge between algebraic theory and geometric visualization.

The mathematical foundation of SVD is deeply rooted in the theory of vector spaces, inner product spaces, orthogonality, eigenvalues, eigenvectors, and matrix diagonalization. Among these concepts, the Spectral Theorem for real symmetric matrices plays a central role because it guarantees the orthogonal diagonalization of symmetric matrices. This theorem provides the theoretical basis for constructing the singular values and singular vectors associated with any matrix through the matrices $A^T A$ and AA^T. Since these matrices are always symmetric and positive semi-definite, they possess complete orthonormal sets of eigenvectors and nonnegative eigenvalues, making the construction of SVD mathematically rigorous.

From a geometric perspective, SVD provides one of the clearest interpretations of linear transformations. Every matrix transformation may be viewed as a sequence consisting of an initial orthogonal rotation (or reflection), a scaling along mutually orthogonal principal directions, and a final orthogonal rotation. Under this interpretation, the unit circle in two-dimensional space or the unit sphere in three-dimensional space is transformed into an ellipsoid whose principal axes correspond to the singular values of the matrix. This geometric interpretation not only explains the behavior of matrices but also clarifies important mathematical concepts such as matrix rank, dimensionality, orthogonality, and principal directions of variation.

Historically, the origins of Singular Value Decomposition can be traced to the independent works of Beltrami and Jordan during the nineteenth century, who introduced early forms of matrix decomposition for classes of matrices. The modern mathematical formulation of SVD was later established through the pioneering work of Eckart and Young (1936), who demonstrated that the decomposition provides the optimal low-rank approximation of a matrix with respect to both the Frobenius and spectral norms. Their work remains one of the cornerstones of modern numerical linear algebra and continues to influence numerous areas of computational mathematics.

The theoretical development of SVD has been further strengthened by numerous researchers. Axler (2015) presented the Spectral Theorem as the principal mechanism for understanding orthogonal diagonalization, thereby providing a rigorous framework for establishing the existence of SVD. Similarly, Strang (2016) emphasized the universality of SVD by demonstrating its applicability to arbitrary rectangular matrices while illustrating its intimate relationship with orthogonality and eigenvalue theory. Horn and Johnson (2013) expanded these theoretical foundations through detailed discussions of matrix analysis, particularly the uniqueness conditions for singular values and singular vectors. Their work explains that while singular values are uniquely determined when properly ordered, singular vectors may not be unique whenever repeated singular values occur.

The geometric interpretation of Singular Value Decomposition has likewise received considerable attention. Lay, Lay, and McDonald (2016) described SVD as a transformation that maps the unit sphere into an ellipsoid through successive rotations and scaling. This visualization provides an intuitive understanding of matrix transformations and demonstrates how singular values determine the lengths of the principal axes while singular vectors define their orientations. Golub and Van Loan (2013) further established the close relationship between singular values, matrix norms, and matrix rank, emphasizing that the Frobenius norm and spectral norm are naturally expressed in terms of singular values. These relationships explain why SVD occupies a central position in numerical linear algebra and matrix computations.

In recent years, the importance of SVD has expanded considerably because of its applications in modern computational science. Raghavendar and Dharmaiah (2017) highlighted its effectiveness in analyzing linear transformations and extracting meaningful structural information from multidimensional data. James et al. (2021) demonstrated its role in Principal Component Analysis, where SVD forms the mathematical foundation for dimensionality reduction while preserving the dominant variation within high-dimensional datasets. More recently, Bezina (2025) showed that SVD provides an efficient mechanism for discovering latent structures, eliminating redundancy, and reducing noise in large datasets, thereby supporting its growing importance in machine learning and data science.

Although the computational applications of Singular Value Decomposition have been extensively studied, much of the existing literature places greater emphasis on numerical algorithms and practical implementations than on the mathematical structures that govern the decomposition itself. In many discussions, the algebraic proofs establishing the existence and uniqueness of SVD are presented independently of their geometric interpretations, making it difficult to appreciate the strong relationship between these two perspectives. Likewise, practical applications are often introduced without sufficiently connecting them to the theoretical principles from which they originate. Consequently, there remains a need for an integrated mathematical treatment that unifies the algebraic foundations, geometric intuition, and computational significance of Singular Value Decomposition within a single theoretical framework.

This study addresses this need by providing a comprehensive mathematical investigation of Singular Value Decomposition in finite-dimensional real vector spaces. Rather than emphasizing algorithmic implementation or software computation, the research focuses on establishing the theoretical foundations that explain why SVD exists, how its uniqueness is characterized, and how its geometric interpretation naturally arises from orthogonal transformations. The study further demonstrates the practical significance of these theoretical concepts through an illustrative application involving student performance data, thereby connecting abstract mathematical theory with a meaningful real-world example.

Specifically, this study aims to (1) define Singular Value Decomposition and derive its formal mathematical construction; (2) establish the algebraic properties and conditions that guarantee the existence and uniqueness of Singular Value Decomposition in finite-dimensional vector spaces; (3) develop a geometric interpretation of Singular Value Decomposition using orthogonal transformations and the concept of ellipsoids; and (4) demonstrate how the mathematical structure of Singular Value Decomposition contributes to solving practical problems through the analysis of multidimensional data.

METHODS

Research Design

This study employed a theoretical mathematical research design to investigate the algebraic and geometric structures underlying Singular Value Decomposition (SVD). Unlike empirical studies that rely on experimental observations or statistical inference, theoretical mathematical research focuses on the formulation, analysis, and rigorous proof of mathematical concepts and relationships. The primary objective was to establish the theoretical foundations of SVD by deriving its mathematical construction, proving its existence and uniqueness, and interpreting its geometric significance within finite-dimensional real vector spaces.

The investigation was grounded in deductive mathematical reasoning, where established definitions, axioms, propositions, and theorems were systematically utilized to derive new results. In particular, the Spectral Theorem for real symmetric matrices served as the principal theoretical framework for proving the existence of Singular Value Decomposition for arbitrary real matrices.

Mathematical Framework

The theoretical framework of this study was built upon fundamental concepts in linear algebra that collectively establish the mathematical basis of Singular Value Decomposition.

The study began with finite-dimensional real vector spaces and inner product spaces, which provide the environment in which vectors, matrices, and linear transformations are defined. Concepts such as vector addition, scalar multiplication, norms, orthogonality, orthonormal bases, and linear independence were first established because they form the structural foundation of matrix decomposition.

The investigation then examined eigenvalues and eigenvectors of real symmetric matrices. Since Singular Value Decomposition is constructed through the eigen structure of the matrices $A^T A$ and AA^T, the Spectral Theorem was employed as the principal analytical tool. The theorem guarantees that every real symmetric matrix possesses an orthonormal basis of eigenvectors and admits orthogonal diagonalization. This property forms the mathematical justification for constructing the singular values and singular vectors associated with any real matrix.

The study also incorporated concepts of matrix norms, matrix rank, orthogonal matrices, diagonal matrices, and linear transformations. These concepts were used to establish the algebraic properties of SVD and to develop its geometric interpretation as a transformation consisting of orthogonal rotations (or reflections) followed by principal-axis scaling.

Mathematical Procedure

Review of Basic Related Concepts

This part established the mathematical concepts necessary for the development of Singular Value Decomposition. Fundamental topics included vector spaces, inner product spaces, orthogonality, orthonormal bases, eigenvalues, eigenvectors, matrix norms, matrix rank, diagonalization, and the Spectral Theorem. These concepts served as the theoretical basis for all succeeding mathematical derivations.

Basic Results

The second part formally defined Singular Value Decomposition and derived its matrix representation $A = U\Sigma V^T$ where U and V are orthogonal matrices and Σ is a diagonal matrix containing the nonnegative singular values of A. The derivation demonstrated how singular values arise from the square roots of the eigenvalues of $A^T A$, while the left and right singular vectors are constructed from the corresponding orthonormal eigenvectors.

Illustrative examples involving low-dimensional matrices were presented to demonstrate the computational procedure and to verify the decomposition.

The next part of the basic results established the theoretical validity of Singular Value Decomposition.

A constructive proof was developed by considering the symmetric positive semi-definite matrix $A^T A$ Applying the Spectral Theorem, the matrix was orthogonally diagonalized to obtain its eigenvalues and orthonormal eigenvectors. The singular values were then defined as the nonnegative square roots of these eigenvalues, while the corresponding left singular vectors were derived directly from the decomposition.

The study further investigated the uniqueness properties of SVD. It was shown that the collection of singular values is uniquely determined when arranged in descending order. Conversely, the singular vectors are unique only when all singular values are distinct; repeated singular values permit infinitely many orthonormal bases within the associated eigenspaces.

Main Results

This part examined the principal algebraic and geometric properties of Singular Value Decomposition. The properties collectively demonstrate the structural significance of SVD in matrix theory and numerical linear algebra. The geometric interpretation was illustrated by examining the transformation of the unit circle in $\mathbb{R}^2$ and the unit sphere in $\mathbb{R}^3$. Graphical representations were employed to demonstrate how these objects are mapped into ellipsoids whose principal axes correspond to the singular values and whose

Although the investigation was primarily theoretical, an illustrative application was included to demonstrate the practical relevance of the mathematical results.

A matrix containing quarterly Mathematics grades of selected Grade 6 students was decomposed using Singular Value Decomposition. The matrices U, Σ, V^T were analyzed to illustrate how SVD identifies dominant patterns, secondary variations, and latent structures within multidimensional educational data.

The application was not intended as an empirical educational study but rather as a mathematical demonstration of how the theoretical principles developed throughout the research can be applied to real-world datasets.

Data Analysis

Since this research was theoretical in nature, no statistical hypothesis testing or inferential statistical procedures were performed. Data analysis consisted entirely of mathematical derivation, theorem proving, algebraic verification, geometric interpretation, and analytical examination of matrix properties. Computational examples were included solely to verify theoretical results and to demonstrate the practical interpretation of Singular Value Decomposition.

The validity of the mathematical results followed directly from established definitions, axioms, and previously proven theorems in linear algebra, particularly the Spectral Theorem for real symmetric matrices. Consequently, the conclusions of the study were established through rigorous deductive reasoning rather than statistical inference.

RESULTS AND DISCUSSION

Singular Value Decomposition

Singular Value Decomposition (SVD) is one of the most powerful matrix factorizations in linear algebra because it exists for every real matrix regardless of its dimensions or rank. Unlike eigenvalue decomposition, which requires a square matrix, SVD provides a universal representation for arbitrary rectangular matrices.

Definition 1

Let

$$A \epsilon \mathbb{R}^{m\times n}$$

A Singular Value Decomposition of A is a factorization of the form

$$A = U\Sigma V^T$$

where

- $U\epsilon\mathbb{R}^{m\times m}$ is an orthogonal matrix,
- $V\epsilon\mathbb{R}^{n\times n}$ is an orthogonal matrix, and
- $\Sigma\epsilon\mathbb{R}^{m\times n}$ is a diagonal matrix whose diagonal entries $\sigma_1, \sigma_2, \dots, \sigma_r$ are called the singular values of A.

The columns of U are called the left singular vectors, while the columns of (V) are called the right singular vectors.

The decomposition expresses every linear transformation as the composition of two orthogonal transformations separated by a diagonal scaling. This representation reveals both the algebraic structure and geometric behavior of matrices.

Existence of Singular Value Decomposition

The principal theoretical result establishes that Singular Value Decomposition exists for every real matrix.

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Theorem 1 (Existence of SVD)

Every real matrix $A \epsilon \mathbb{R}^{m \times n}$ has a Singular Value Decomposition.

Proof:

Let's start by considering the matrix $A^T A$. Let $A \in \mathbb{R}^{m \times n}$. Then we can say that $A^T A \in \mathbb{R}^{n \times n}$ is symmetric and positive semi-definite because for any $x \in \mathbb{R}^n$,

$$x^T A^T A x = (Ax)^T (Ax) = \|Ax\|^2 \geq 0.$$

By using the spectral theorem and since $A^T A$ is real and symmetric, then we can say that it has a complete set of orthonormal eigenvectors where in there exists an orthogonal matrix $V \in \mathbb{R}^{n \times n}$ such that,

$$A^T A = V \Lambda V^T$$

where $\Lambda = diag(\lambda_1, \dots, \lambda_n)$, and λ_i are non-negative eigenvalues of $A^T A$.

We know that the set $\sigma_i = \sqrt{\lambda_i} \geq 0$ for $i = 1, \dots, n$ are the singular values of A. Let's define a matrix $\Sigma \in \mathbb{R}^{m \times n}$ where the σ_i is on the diagonal for $i = 1, \dots, r$ and $r = rank(A)$ and the remaining entries will be 0.

It will define $\Sigma \in \mathbb{R}^{m \times n}$ as

$\Sigma_{ii} = \sigma_i$ for $i = 1, \dots, r$, $\Sigma_{ij} = 0$ otherwise.

Let $v_i, \dots, v_n \in \mathbb{R}^n$ the orthonormal eigenvectors of $A^T A$, for which will form the columns of V, such that

$A^T A v_i = \lambda_i v_i$, for $i = 1, \dots, n$

For each $i \in \{1, \dots, r\}$ such that $\sigma_i > 0$, we can define

$$u_i = \frac{1}{\sigma_i} A v_i \in \mathbb{R}^m$$

where v_i is the i-th column of V. We can say now that for each $u_i \in \mathbb{R}^m$, it will satisfy

$$\|u_i\|^2 = \left\| \frac{1}{\sigma_i} A v_i \right\|^2 = \frac{1}{{\sigma_i}^2} \|Av_i\|^2 = \frac{1}{{\sigma_i}^2} {v_i}^T A^T A v_i = \frac{1}{{\sigma_i}^2} {v_i}^T (\lambda_i v_i) = \frac{\lambda_i}{{\sigma_i}^2} = 1$$

Therefore, u_i is a unit vector of AA^T.

Furthermore, for $i \neq j$,

$$\langle u_i, u_j \rangle = \frac{1}{\sigma_i \sigma_j} \langle A v_i, A v_j \rangle = \frac{1}{\sigma_i \sigma_j} {v_i}^T A^T A v_j = \frac{1}{\sigma_i \sigma_j} {v_i}^T \left(\lambda_j v_j \right) = \frac{\lambda_j}{\sigma_i \sigma_j} \langle v_i, v_j \rangle = 0$$

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Since $v_i \perp v_j$. Therefore, $u_i, \ldots, u_r$ are orthonormal. By using Gram-Schmidt process, the set $\{u_i, \ldots, u_r\}$ will form to an orthonormal basis $\{u_1, \ldots, u_m\}$ for $\mathbb{R}^{m\times m}$.

This will define the orthogonal matrix $U = [u_1, \ldots, u_m] \in \mathbb{R}^{m\times m}$. Let's now verify

$$A = U\Sigma V^T$$

Since

i. $U = [u_1, \ldots, u_m] \in \mathbb{R}^{m\times m}$, where $u_1, \ldots, u_r$ are defined as $u_i = \frac{1}{\sigma_i} A v_i$ (u_i is a unit vector of AA^T) and the rest complete the basis and U is an orthogonal matrix.

ii. $V = [v_1, \ldots, v_n] \in \mathbb{R}^{n\times n}$, where the columns are the eigenvectors of $A^T A$ and V is an orthogonal matrix.

iii. $\Sigma \in \mathbb{R}^{m\times n}$ such that

$$\Sigma_{ij} = \begin{cases} \sigma_i, & if\ i = j \leq r \\ 0, & otherwise \end{cases}$$

Then

$$U\Sigma V^T = \sum_1^r \sigma_i u_i {v_i}^T$$

$$= \sigma_1 u_1 {v_1}^T + \sigma_2 u_2 {v_2}^T + \cdots + \sigma_r u_r {v_r}^T$$

Recall that

$$u_i = \frac{1}{\sigma_i} A v_i$$

Then

$A v_i = \sigma_i u_i$ for$i = 1, \ldots, r$

So

$$A = \sum_1^r A v_i {v_i}^T = \sum_1^r \sigma_i u_i {v_i}^T$$

Hence,

$A = U\Sigma V^T$ ■

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This completes the proof.

The theorem confirms that every real matrix possesses an SVD regardless of its dimensions. This universality distinguishes SVD from many other matrix factorizations and explains its extensive use throughout mathematics and computational science.

Uniqueness of Singular Values and Singular Vectors

Although every matrix possesses a Singular Value Decomposition, the uniqueness of its components differs.

Theorem 2 (Uniqueness of Singular Values)

Let $A \in \mathbb{R}^{m\times n}$. The singular values of A are uniquely determined, up to ordering.

Proof: The singular values of A are the nonnegative square roots of eigenvalues of the matrix $A^T A$, that is:

$$\sigma_i = \sqrt{\lambda_i},$$

where λ_i are the eigenvalues of $A^T A$.

Since $A^T A$ is a symmetric matrix, its eigenvalues are uniquely determined (up to ordering). Therefore, the singular values σ_i are also uniquely determined (up to ordering). ■

Theorem 3 (Uniqueness of Singular Vectors)

Let

$$A = U\Sigma V^T$$

be the Singular Value Decomposition of A.

1. If all singular values are distinct, then the corresponding singular vectors are unique up to sign.
2. If some singular values are repeated, then the corresponding singular vectors are not unique, but any orthonormal basis of the associated subspace is valid.

Proof: Consider the matrix $A^T A$, which is symmetric and admits an orthonormal set of eigenvectors.

Case 1: If the eigenvalues of $A^T A$ are distinct, then the corresponding eigenvectors are unique up to scalar multiples. Since the eigenvectors are normalized in SVD, the only remaining ambiguity is multiplication by ± 1.

Thus, the right singular vectors (columns of V) are unique up to sign. The left singular vectors (columns of U), defined by

$$u_i = \frac{1}{\sigma_i} A v_i,$$

inherit the same uniqueness property.

Case 2: If some eigenvalues are repeated, then the corresponding eigenspace has dimension greater than one. In this case, there are infinitely many choices of orthonormal bases for that eigenspace.

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Thus, the singular vectors are not unique, but any orthonormal basis of the eigenspace yields a valid SVD.

Therefore,

The SVD of a matrix $A \in \mathbb{R}^{m \times n}$ is unique for the following conditions:
1. The singular values in the ordering $\sigma_1 \geq \sigma_2 \geq \cdots \sigma_r \geq 0$, then the set $\{\sigma_i\}$ is unique.
2. The columns of U and V which is the left and right singular vectors is repeated or in the multiplicity of greater than 1.

Computational Algorithm

The theoretical construction naturally leads to an algorithm for computing the Singular Value Decomposition.

Let A be an $m \times n$ real matrix. The singular value decomposition of A is given by:

$$A = U\Sigma V^T$$

where:
- U is an $m \times m$ orthogonal matrix,
- V is an $n \times n$ orthogonal matrix,
- Σ is an $m \times n$ diagonal matrix with nonnegative entries.

The SVD may be computed through the following steps:
1. Compute $A^T A$
2. Solve for eigenvalues and construct matrix Σ.
3. Compute the right singular vectors (Matrix V)
4. Compute the left singular vectors (Matrix U)
5. Construct the final SVD form.

Modern numerical algorithms, such as the Golub–Kahan bidiagonalization method, improve computational efficiency for large-scale matrices. Nevertheless, the theoretical procedure presented above remains the foundation for understanding the mathematical construction of SVD.

Geometric Interpretation of Singular Value Decomposition

One of the principal findings of this study is that Singular Value Decomposition provides a complete geometric description of every linear transformation in finite-dimensional Euclidean space. The decomposition demonstrates that any real matrix can be interpreted as a sequence of three elementary geometric operations: an initial orthogonal rotation (or reflection), a scaling along mutually orthogonal principal directions, and a final orthogonal rotation (or reflection). This result establishes a direct relationship between the algebraic representation of a matrix and its geometric action on vectors.

Let

$$A = U\Sigma V^T$$

where U and V are orthogonal matrices and Σ is a diagonal matrix of singular values. The study shows that the matrix V^T first rotates or reflects the coordinate system so that it aligns with the right singular vectors.

This transformation changes only the orientation of the coordinate axes while preserving vector lengths, angles, and orthogonality.

The diagonal matrix Σ then scales the transformed vectors independently along the orthogonal coordinate axes. Each singular value determines the magnitude of stretching or compression in its corresponding principal direction. Larger singular values indicate directions of greater variation, whereas smaller singular values correspond to directions that experience less stretching. Singular values equal to zero collapse their corresponding dimensions entirely, thereby reducing the effective rank of the transformation.

Finally, the orthogonal matrix U performs another rotation (or reflection), orienting the scaled vectors into their final positions in the output space. Since orthogonal transformations preserve distances and angles, the only change in magnitude occurs during the intermediate scaling stage.

Consequently, the study establishes that every linear transformation may be interpreted geometrically as a Rotation–Scaling–Rotation process,

$$A = Rotation \rightarrow Scaling \rightarrow Rotation$$

which provides a unified geometric interpretation of matrix multiplication.

A significant result of the investigation is the demonstration that the image of the unit circle in $\mathbb{R}^2$, or the unit sphere in $\mathbb{R}^3$, under the transformation A is always an ellipsoid. The lengths of the semi-principal axes of the ellipsoid are precisely the singular values of the matrix, while the orientations of these axes are determined by the corresponding left singular vectors. Thus, the singular values quantify the amount of stretching in each principal direction, whereas the singular vectors identify the directions in which these transformations occur.

The geometric interpretation also provides a natural explanation for several important algebraic properties of matrices. First, the number of nonzero singular values equals the rank of the matrix, implying that rank corresponds to the number of independent directions preserved after transformation. Second, the largest singular value represents the maximum stretching factor of the transformation and is therefore equal to the spectral norm of the matrix. Third, when one or more singular values are zero, the corresponding dimensions are mapped to lower-dimensional subspaces, thereby explaining rank deficiency in purely geometric terms.

These findings demonstrate that Singular Value Decomposition not only decomposes matrices algebraically but also reveals their intrinsic geometric behavior. The decomposition transforms an otherwise complex linear mapping into a sequence of simple geometric operations that are intuitive to visualize and mathematically rigorous to analyze. This unified interpretation strengthens the connection between abstract linear algebra and geometric intuition, making concepts such as orthogonality, rank, matrix norms, and dimensionality easier to understand within a single mathematical framework.

The results obtained in this study are consistent with the geometric interpretations presented by Lay, Lay, and McDonald (2016), Strang (2016), and Golub and Van Loan (2013), while extending these classical discussions by integrating the algebraic proofs of existence and uniqueness with the geometric visualization of linear transformations. This unified treatment demonstrates that the theoretical and geometric perspectives of Singular Value Decomposition are complementary rather than independent, providing a comprehensive understanding of one of the most fundamental factorizations in linear algebra.

DISCUSSION

The results presented in this section establish the fundamental mathematical properties of Singular Value Decomposition. The existence theorem demonstrates that SVD applies universally to every real matrix, while the uniqueness theorem clarifies the conditions under which its components are uniquely determined. Together, these results provide the theoretical basis for interpreting SVD as both an algebraic factorization and a geometric

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transformation. These findings are consistent with the classical treatments of matrix analysis presented by Eckart and Young (1936), Horn and Johnson (2013), Golub and Van Loan (2013), and Strang (2016), all of whom recognize SVD as one of the central tools of modern linear algebra. The theoretical framework developed in this section serves as the basis for the succeeding analysis of algebraic properties, geometric interpretation, and practical applications of Singular Value Decomposition.

CONCLUSION

This study investigated the algebraic and geometric structures underlying Singular Value Decomposition (SVD) and demonstrated its significance as one of the most fundamental matrix factorizations in linear algebra. Through rigorous mathematical analysis, the study established that every real matrix admits a Singular Value Decomposition, with its existence guaranteed by the Spectral Theorem through the orthogonal diagonalization of the symmetric positive semi-definite matrix $A^T A$. The findings further confirmed that the singular values are uniquely determined when arranged in descending order, whereas the corresponding singular vectors are unique only when the singular values are distinct, with repeated singular values permitting multiple orthonormal bases within their associated eigenspaces.

From a geometric perspective, the study demonstrated that SVD provides a complete interpretation of linear transformations as a sequence of orthogonal rotation (or reflection), diagonal scaling, and a final orthogonal rotation. This Rotation–Scaling–Rotation framework shows that every linear transformation maps the unit circle or unit sphere into an ellipsoid whose semi-principal axes are determined by the singular values and whose orientations are determined by the singular vectors. Consequently, SVD offers an intuitive geometric explanation of important matrix properties, including matrix rank, orthogonality, dimensionality, and matrix norms, thereby bridging abstract algebraic concepts with geometric visualization.

The study also established that SVD provides a powerful mathematical framework for understanding the internal structure of matrices through their rank-one decomposition. This representation forms the theoretical basis for low-rank approximation, enabling the identification of dominant components while reducing redundant or insignificant information. The illustrative application involving student performance data demonstrated that SVD effectively extracted dominant learning trends, secondary variations, and latent structures within a multidimensional dataset, thereby illustrating how theoretical mathematical concepts can be applied to practical problems in educational data analysis.

Overall, the findings confirm that Singular Value Decomposition is not merely a computational algorithm but a comprehensive mathematical framework that unifies algebraic theory, geometric intuition, and practical computation. Its universal applicability to arbitrary matrices, together with its ability to reveal the intrinsic structure of linear transformations, makes SVD an indispensable tool in modern linear algebra. Beyond its theoretical importance, the decomposition provides the mathematical foundation for numerous applications, including dimensionality reduction, principal component analysis, signal processing, image compression, machine learning, and scientific computing. As such, this study contributes to a deeper understanding of Singular Value Decomposition by integrating its theoretical foundations, geometric interpretation, and practical significance into a unified mathematical perspective.

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