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On Subset Labeling of Some Classical Graph Families

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ABSTRACT

Let $G = (V(G), E(G))$ be a nontrivial connected simple graph. For $[r] = \{1,2,\dots,r\}$, let $P^*[r] = P[r]\backslash\{\emptyset\}$ denote the family of all nonempty subsets of $[r]$. A subset $r-$labeling of G is a vertex labeling $f: V(G) \to P^*[r]$ such that for any distinct vertices $u, v \epsilon V(G)$, the edge $uv \epsilon E(G)$ if and only if $f(u) \cap f(v)$. The subset index of G, denoted $\rho(G)$, is the minimum integer r such that a subset $r-$labeling of G exists.

In this paper, we investigate the behavior of the subset index and construct valid subset labelings for several classical graph families. In particular, we determine the subset index of crown, pan, wheel, and gear graphs, and provide explicit constructions of their proper subset labelings.

Keywords: *Subset Labeling, Subset Index, Vertex Labeling, Crown Graph, Pan Graph, Wheel Graph, Gear Graph*

INTRODUCTION

Subset labeling is a graph-theoretic framework in which adjacent vertices are assigned disjoint labels and non-adjacent vertices are assigned intersecting labels. Foundational work has established subset indices for paths, cycles, trees, and complete graphs positioning the concept as a distinct extension of classical labeling methods. In particular, the subset indices of complete graphs, paths, and cycles serve as critical benchmarks for further development. Building on these foundations, the present study extends subset labeling to classical graph families—specifically crown, pan, wheel, and gear graphs—constructing valid labelings and determining their indices. In doing so, the study expands the theoretical landscape of subset labeling by applying it to new structures and broadening its applicability within graph theory.

METHODS

This study employs expository, descriptive, and constructive approaches to investigate subset labeling in classical graph families. The expository component introduces definitions, notation, and established results to provide a clear theoretical foundation. The descriptive analysis examines the structural properties of crown, pan, wheel, and gear graphs, highlighting labeling rules and emerging patterns. The constructive method develops explicit subset labeling assignments for representative cases, verifying theoretical results and illustrating general strategies. Together, these methods form a rigorous framework for analyzing subset labeling in classical families.

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RESULTS

This section presents the subset indices of the crown, pan, wheel, and gear graphs together with explicit constructions of their proper subset labelings.

Crown Graph

Definition 1. A crown graph $H_{n,n}$ is obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching. Its vertex set is $V(H_{n,n}) = \{y_1, y_2, \ldots, y_n\} \cup \{z_1, z_2, \ldots, z_n\}$, and its edge set is $E(H_{n,n}) = \{(y_i, z_j) \mid 1 \le i, j \le n, i \ne j\}$. Thus, each y_i is adjacent to all z_j except z_i.Crown graph has $2n$ vertices and $n(n-1)$ edges.

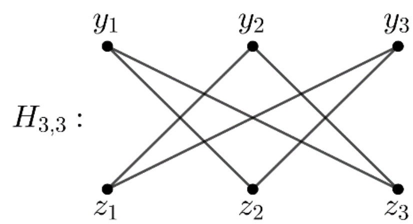

Figure 1: *Illustration of the graph* $H_{3,3}$.

Theorem 1. For any crown graph $H_{n,n}$, the subset index satisfies $\rho(H_{n,n}) = n + 2$.

Subset labeling construction

$$g(x) = \begin{cases} \{i, n+1\}, & x = y_i, \\ \{i, n+2\}, & x = z_i \end{cases}$$

where $\{y_1, y_2, \ldots, y_n\}$ and $\{z_1, z_2, \ldots, z_n\}$ are the two independent partite sets of $H_{n,n}$.

Pan Graph

Definition 2. A pan graph Pan_n is obtained by attaching a single pendant vertex to a cycle graph C_n, where $n \ge 3$. Its vertex set is $V(Pan_n) = \{u_1, u_2, \ldots, u_n, v\}$, where v is the pendant vertex attached to u_n. Its edge set is $E(Pan_n) = \{(u_i, u_{i+1}) \mid 1 \le i \le n-1\} \cup \{(u_n, u_1)\} \cup \{(u_n, v)\}$. Thus, it resembles a cycle with a single attached vertex and has $n + 1$ vertices and $n + 1$ edges.

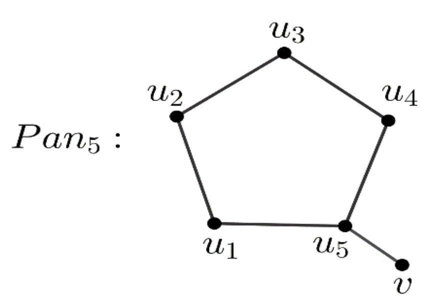

Figure 2: *Illustration of the graph* Pan_5.

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Theorem 2. For any pan graph Pan_n, the subset index satisfies $\rho(Pan_n) = \rho(C_n),\ n \neq 4$.

Subset labeling construction

$$g(x) = \begin{cases} f(x), & x \in V(C_n), \\ [r] \backslash f(u_n), & x = v \end{cases}$$

where $f : V(C_n) \to P([r])$ is a subset labeling of C_n, v is the pendant vertex, and u_n is its unique neighbor in C_n.

Theorem 3. For the cycle C_4, the subset index of the pan graph differs: $\rho(C_4) = 2$ and $\rho(Pan_4) = 3$.

Subset labeling construction

$$g(x) = \begin{cases} f(x), & x \in V(C_4) \backslash \{u_2\}, \\ f(u_2) \cup \{3\}, & x = u_2, \\ (V(C_4) \backslash f(u_n)) \cup \{3\}, & x = v \end{cases}$$

where f is a valid subset labeling of C_4 and $\rho(C_4) = 2$, v is the pendant vertex attached to u_n, and u_2 is the vertex opposite u in C_4.

Wheel Graph

Definition 3. A wheel graph W_n is formed by joining a universal vertex to all vertices of a cycle C_n, where $n \geq 3$. Its vertex set is $V(W_n) = \{u_1, u_2, \dots, u_n, w\}$, where w is the central hub, and its edge set is $E(W_n) = \{(u_i, u_{i+1}) \mid 1 \leq i \leq n-1\} \cup \{(u_n, u_1)\} \cup \{(w, u_i) \mid 1 \leq i \leq n\}$. Thus, W_n has $n+1$ vertices and $2n$ edges.

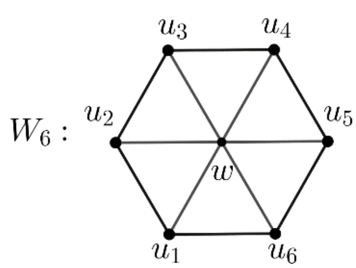

Figure 3: *Illustration of the graph W_6.*

Theorem 4. Let W_n be the wheel graph on n vertices, obtained by joining a cycle C_n to a universal vertex w. Then $\rho(W_n) = \rho(C_n) + 1$

Subset labeling construction

$$g(x) = \begin{cases} f(x), & x \in V(C_n), \\ \{r+1\}, & x = w \end{cases}$$

where $f : V(C_n) \to P([r])$ is a subset labeling of C_n and w is the central vertex.

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Gear Graph

Definition 4. A gear graph G_n is obtained by subdividing each edge of the cycle in a wheel graph W_n. Its vertex set is $V(G_n) = V(W_n) \cup \{v_i \mid 1 \leq i \leq n\} \cup \{w\}$, and its edge set is $E(G_n) = \{(v_i, u_i), (u_i, v_{i+1}) \mid 1 \leq i \leq n-1\} \cup \{(v_n, u_n), (u_n, v_1)\} \cup \{(w, u_i) \mid 1 \leq i \leq n\}$. Thus, G_n has $2n+1$ vertices and $3n$ edges, and can be viewed as a wheel graph with additional vertices inserted along the cycle edges.

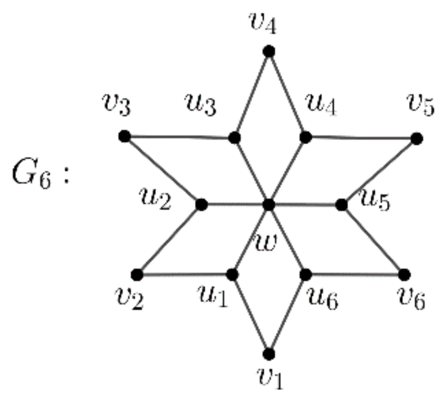

Figure 4: *Illustration of the graph G_6.*

Theorem 5. For the gear graph G_n with $3 \leq n \leq 9$, $\rho(G_n) = \rho(C_{2n})$.

Subset labeling construction

$$g(x) = \begin{cases} f(u_i), & x = u_i, i = 1,2, \ldots n, \\ f(v_i), & x = v_i, i = 1,2, \ldots n \\ \bigcap_{i=1}^{n} f(v_i), & x = w \end{cases}$$

where $f : V(C_{2n}) \rightarrow P([r])$ is an optimal subset labeling of C_{2n}, whose vertices are labeled consecutively as $v_1, u_1, v_2, u_2, \ldots, v_n, u_n$, and w is the universal vertex adjacent to all u_i.

Remark. The restriction $3 \leq n \leq 9$ reflects the range where explicit labelings of C_{2n} extend to G_n without introducing new elements for the universal vertex w. For larger n, the labeling of C_{2n} becomes more intricate, and it is not yet clear whether such an extension preserves all adjacency and non-adjacency conditions.

CONCLUSION

Classical graph families are distinguished by their intrinsic structures, whether through bipartite arrangements, central hub formations, or cycle-based configurations. The subset labelings of paths, cycles, and complete graphs provide the foundational basis for these constructions, serving as building blocks from which more complex subset assignments emerge. By analyzing pan, crown, wheel, and gear graphs, this study demonstrates how classical structures inherit and extend labeling patterns derived from simpler graphs. This progression from foundational graphs to classical families enables a comparative understanding of how subset indices behave across inherently structured graph classes. The findings reinforce the consistency of the subset labeling framework across diverse graph structures while illustrating how the structural properties of individual graph families influence their subset labeling constructions and subset indices.

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