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On the Jones Polynomial of Knot Types

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ABSTRACT

In the recent years, knot theory has become a tool in applied mathematics, covering knots in higher dimensions by considering n-dimensional spheres and m-dimensional Euclidean space. While mathematical knots in general can be broad-ranging, narrowing down concepts can create interesting subject matters that may help understanding knots in a more comprehensive way, for instance, knot invariants. These invariants are quantities defined for each knot. One of the most known and significant knot invariants is the Jones polynomial. For certain types of knot diagrams, such as trefoil knot, figure-eight knot, and cinquefoil knot, there are various efficient classical algorithms for evaluating their Jones polynomial. This paper explores the concept of Jones polynomial in the mathematical field of knot theory and investigates how it is assigned to a particular knot. In addition, it seeks to identify properties of Jones polynomial that are significant to the knot types utilized in this study. It also discusses its relationship to other topological theories and invariants such as Chern-Simons theory, quantum knot invariants, and Khovanov homology. The finding establishes that for the first knot types with up to five crossings, the Jones polynomial can be generated by applying the skein relation. Overall, the study contributes to the systematic presentation and deeper understanding of Jones polynomial as a significant tool in studying mathematical knots.

Keywords: *Jonel polynomial, knot theory, trefoil knot, figure-eight knot, cinquefoil knot, Chern-Simons theory, quantum knot invariants, Khovanov homology.*

INTRODUCTION

In topology, a branch of mathematics that studies properties of shapes that are preserved under continuous deformations, such as stretching, crumpling, and bending, but not tearing or gluing, there is a mathematical framework that studies knots called knot theory. While inspired by knots which appear in daily life, such as those in shoelaces and rope, a mathematical knot differs in that the ends are joined so it cannot be undone. In mathematical language, a knot is an embedding of a circle in 3-dimensional Euclidean space.

The study of knots can be given some motivation in terms of applications in molecular biology or by reference to parallels in equilibrium statistical mechanics or quantum field theory. Here, however, knot theory is considered as part of geometric topology. Motivation for such a topological study of knots is meant to come from a curiosity to know how the geometry of three-dimensional space can be explored by knotting phenomena using precise mathematics.

One way of distinguishing knots is by using a knot invariant, a "quantity" which remains the same even with different descriptions of a knot. The concept of a knot has been extended to higher dimensions by considering n-dimensional spheres in m-dimensional Euclidean space. This was investigated most actively in the period 1960-1980, when a number of breakthroughs were made. In recent years, low dimensional phenomena have garnered the most interest. Research in knot theory began with the creation of knot tables and the systematic tabulation of

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knots. While tabulation remains an important task, today's researchers have a wide variety of backgrounds and goals. In the recent years, knot theory has also become a tool in applied mathematics.

The Jones polynomial, an important knot invariant, is known to be of exponential time on the number of crossings of a knot diagram. For special types of knot diagrams, such as torus knots, algebraic links, pretzel links, 2-bridge links, link diagrams with bounded treewidth, 3-braid links and Montesinos links, there are various efficient classical algorithms for evaluating the Jones polynomial. Even though important for our understanding of knots, these remain special cases of knots, and, in practice, such cases may have very low probability of occurrence.

Thus, this paper explores the concept of Jones polynomial in mathematical field of knot theory and how it is assigned to a particular knot. Specifically, it discusses its definition, major properties, and relationships with Chern-Simons theory, quantum knot invariants, and Khovanov homology. Through a descriptive-expository approach, the paper presents the Jones polynomial as a significant mathematical tool in understanding how mathematical knots work.

METHODS

This study employed a descriptive-expository approach. This means that this study was based purely on existing information while also investigating new concepts and ideas that are backed up by facts and evidence

The first part of the study builds the theoretical conceptual background needed for understanding the Jones polynomial of certain knot types. It begins with the basic concepts in knot theory such as defining the different types of knots, the knot invariants other than the Jones polynomial, and some mathematical tools to be used to get the Jones polynomial of knots. These ideas are important because they form the basis for analyzing the Jones polynomials in knot theory.

The next phase of the study includes the generation of the Jones polynomials of the knot types cited in the scope and limitations of this research. It utilizes mathematical tools such as the skein relation and the Reidemeister moves in bringing about the polynomial. This part uses precise mathematical proofs in constructing how Jones polynomials are generated.

Following that, the study moves on citing the key properties of the Jones polynomial by proving their existence.

The last part of the study focuses on the relationship of Jones polynomial to the other topological theories such as the Chern-Simons theory, quantum knot invariant, and the Khovanov homology, which are all freestanding on their own essence.

Although this study is purely descriptive and theoretical, it provides basic understanding of how Jones polynomial works and what is its significance in the field of knot theory.

RESULTS AND DISCUSSION

The Jones Polynomial

The Jones polynomial $V_K(t)$ is defined as a Laurent polynomial in the variable $\sqrt{t}$ which is defined for every oriented knot K and link L but depends on that link up to orientation preserving diffeomorphism, or equivalently isotopy, of $\mathbb{R}^3$.

It is defined by the axioms:

i. If K and K' are equivalent as oriented links, then $V_K(t) = V_{K'}(t)$.
ii. If U is the unknot, then $V_U(t) = 1$.
iii. If K_+, K_-, and K_0 are oriented knots related by the standard skein relation, then we have the following quantity:

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$$t^{-1}V_{K_+}(t) - tV_{K_-}(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V_{K_0}(t)$$

The Jones polynomial for certain knot types was generated by applying the skein relation and performing the Reidemeister moves:

1.1 The unknot has a Jones polynomial of 1.
1.2 For a 3_1 knot, $V_{3_1}(t) = t + t^3 - t^4$.
1.3 For a 4_1 knot, $V_{4_1}(t) = t^{-2} - t^{-1} + 1 - t + t^2$.
1.4 For a 5_1 knot, $V_{5_1}(t) = t - t^2 + 2t^3 - t^4 + t^5 - t^6$.

Key Properties of Jones Polynomial

The following are the key properties of Jones polynomial:

1.1 The Jones polynomial is an ambient isotopy invariant of oriented links satisfying axioms i, ii, iii, and is uniquely determined by these axioms.
1.2 The Jones polynomial is an invariant of oriented links.

Relationship with Other Topological Theories

Another major result of the study is the relationship of Jones polynomial to several topological theories and invariants related to it.

Connection to Chern-Simons Theory

The Jones Polynomial of a given knot can be obtained by considering Chern-Simons theory on the three-sphere with gauge group $SU(2)$, and computing vacuum expectation value of a Wilson loop associated to a knot, and the fundamental representation F of $SU(2)$.

By definition, Chern-Simons theory has an oriented 3-manifold M, and a compact single gauge group G.

Connection to Quantum Knot Invariants

Quantum knot invariants are powerful numerical invariants defined by Quantum Field theory with deep connections to the geometry and topology in dimension three. This is a survey talk on the various limits and colored Jones polynomial, one of the best-known quantum knot invariants. This is a 25-year-old subject that contains theorems and conjectures in disconnected areas of mathematics.

Connection to Khovanov Homology

In search of a stronger knot invariant than the Jones polynomial, Khovanov develops a way to make an oriented link diagram, L, and transform the diagram into a bi-graded chain complex, $C^{*,*}(L)$, the homology of which is the Khovanov homology of $L, H(L)$. In the form of a diagram:

$$L \xrightarrow{Khovanov} C^{*,*}(L) \xrightarrow{homology} \mathcal{H}(L)$$

CONCLUSION

This study concludes that Jones polynomial is one of the most important knot invariants that is known to be of exponential time on the number of crossings of a knot diagram as this polynomial tell knots apart from each other. The existence of the polynomial for distinct knots is guaranteed by the properties which serve as the foundation for how these polynomials are generated, allowing consistent results. Furthermore, several topological theories and related invariants are also bridged to Jones polynomials to see how they can relate to each other.

References

Adams, C. (2004). *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots.* American Mathematical Society.

Amankwah, D. (2014). *The Jones polynomial and its limitations*. arXiv. https://arxiv.org/abs/1407.2196

Barkataki, K., & Panagiotou, E. (2025). *A parallel algorithm for the computation of the Jones polynomial*. arXiv. https://arxiv.org/abs/2505.23101

Burde, G., Zieschang, H., & Heusener, M. (2014). *Knots*. *3*. De Gruyter.

Charbonneau, C. (2021). *Knots and knot invariants: an introduction*. University of Vermont.

Chen, Y. & Shrock, R. (2025). *Jones polynomial and their zeros for a family of knots and links.* arXiv. https://arxiv.org/abs/2507.03680v2

Dimitrov, D. & Patterson, I. (2021). *Invariants in knot theory*. MIT Mathematics.

Dlotko, P., Gurnari, D., & Sazdanovic, R. (2021). *Knot invariants and their relations: a topological perspective*. arXiv. https://arxiv.org/abs/2109.00831v1

Eagles, N. (2023). *Unknotting knots and the Jones polynomial*. University of California at Berkeley.

Fivaable. (n.d.). *Knot theory basics: intro and definitions.* https://fiveable.me/knot-theory/unit-1

Garoufalidis, S. (2013). *Quantum knot invariants*. arXiv. https://arxiv.org/abs/1201.3314

Ghandi, K. (2013). *Khovanov homology as an invariant*. Semantic Scholar.

Jones, V. (2005). *The Jones polynomial*. University of California at Berkeley.

Lickorish, R. (1997). *An Introduction to Knot Theory*. Springer Science & Business Media.

Liu, H. (2022). *A brief introduction to knot theory and the Jones polynomial*. Semantic Scholar.

Livingston, C. (1993). *The Carus Mathematical Monographs*. *24*. American Mathematical Society.

Lorton, C. (2009). *On the breadth of the Jones polynomial for certain classes of knots and links*. Western Kentucky University.

Mifsud, P. (2018). Chern-Simons Theory. *Knot Theory, The Jones Polynomial and Chern-Simons Theory*. 36-40.

Phillips, T. (n.d.). *Knots and their polynomials.* Stony Brook University.

Science Daily. (n.d). *Knot theory.* https://www.sciencedaily.com/terms/knot_theory.htm

Sleiman, J., et al. (2023). *Geometric learning of knot topology.* https://doi.org/10.1039/D3SM01199B

Steinhauer, H. (2020). *An analysis of comparison of knot polynomials*. James Madison University.

The Math in Moscow. (n.d.). *Lecture 4. The Jones polynomial.* https://mathinmoscow.org/wp-content/uploads/KnotsF21Lecture4

Whitman College. (n.d.). *Introduction to higher mathematics.* https://www.whitman.edu/mathematics/higher_math_online/section02.01.html