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On Triangle of Narayana Numbers

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ABSTRACT

This paper investigates the Triangle of Narayana Numbers as a refined combinatorial structure arising from Dyck paths classified according to the number of peaks. Using a theoretical-deductive and expository approach, the study establishes the definition and closed-form expression of the Narayana numbers, given by

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1},$$

where $N(n,k)$counts the number of Dyck paths of semilength $n$with exactly $k$peaks. The paper further examines the major algebraic and structural properties of the Narayana triangle, including boundary values, symmetry, recurrence relations, determinant and quadratic identities, central column formulas, parity patterns, operator representations, partial sums, and generating functions. In addition, it explores the relationship of the Narayana numbers with Catalan numbers, Pascal's triangle, and unsigned Lah numbers, showing how these connections enrich their role in enumerative combinatorics. Selected applications are also discussed, particularly in noncrossing set partitions, unlabeled ordered rooted trees, and 132-avoiding permutations classified by descents. The findings affirm that the Narayana triangle is not merely a refinement of Catalan counting but a mathematically significant structure with broad combinatorial interpretations. Overall, the study contributes to the systematic presentation and deeper understanding of Narayana numbers as an important object in discrete mathematics and enumerative combinatorics.

Keywords: *Narayana numbers, Narayana triangle, Dyck paths, Catalan numbers, enumerative combinatorics, noncrossing partitions, rooted trees, 132-avoiding permutations.*

INTRODUCTION

Enumerative combinatorics is a significant branch of discrete mathematics concerned with counting, arranging, and analyzing finite structures. It provides systematic methods for studying mathematical objects such as sequences, partitions, permutations, trees, and lattice paths. Among the important structures in this field are triangular arrays and counting sequences, which often reveal meaningful patterns, recurrence relations, and combinatorial interpretations.

One of the classical sequences in combinatorics is the Catalan sequence, which appears in many counting problems involving Dyck paths, rooted trees, polygon triangulations, and noncrossing partitions. However, while Catalan numbers count such structures according to size, the Narayana numbers provide a more refined classification by considering additional structural features. In particular, Narayana numbers classify Dyck paths according to the number of peaks, giving rise to the Triangle of Narayana Numbers.

The Triangle of Narayana Numbers is not only a counting arrangement but also a rich mathematical structure. Its entries exhibit symmetry, boundary behavior, recurrence relations, generating functions, parity patterns, and connections with other well-known combinatorial arrays. The row sums of the Narayana triangle are

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related to Catalan numbers, showing that the Narayana numbers serve as a refinement of Catalan enumeration. Moreover, their connections with Pascal's triangle and unsigned Lah numbers further demonstrate their importance in the broader study of discrete mathematical structures.

Despite the established significance of Narayana numbers in enumerative combinatorics, their properties, relationships, and applications remain valuable areas for systematic mathematical exposition. A clearer investigation of the Triangle of Narayana Numbers may help explain its role in counting Dyck paths, noncrossing set partitions, unlabeled ordered rooted trees, and 132-avoiding permutations. Such discussion strengthens the understanding of how one combinatorial structure may connect several counting problems in discrete mathematics.

Thus, this paper investigates the Triangle of Narayana Numbers arising from peaks in Dyck paths. Specifically, it discusses its definition, major properties, relationships with Catalan numbers, Pascal's triangle, and unsigned Lah numbers, and selected applications in enumerative combinatorics. Through a theoretical-deductive and expository approach, the paper presents the Narayana triangle as a significant mathematical object whose structural richness extends beyond its role as a refinement of Catalan counting.

METHODS

This study employed a theoretical-deductive research design using an expository mathematical approach. This design was appropriate because the study focused on the Triangle of Narayana Numbers as an abstract mathematical structure and did not require empirical data gathering, surveys, experiments, or statistical treatment.

The investigation was carried out through logical reasoning, algebraic derivation, combinatorial analysis, and proof-based discussion. Established definitions and known combinatorial concepts were used as foundations in examining the properties, relationships, and applications of the Narayana triangle. The study also utilized examples and illustrations to clarify how the Narayana numbers arise from Dyck paths and how they relate to other combinatorial structures.

As an expository mathematical study, the paper organized the discussion through definitions, propositions, theorems, proofs, and examples. This approach allowed the results to be presented in a systematic and coherent manner, emphasizing both the structural properties and combinatorial significance of the Triangle of Narayana Numbers.

RESULTS AND DISCUSSION

Narayana Numbers and Dyck Path Interpretation

The study established that the Triangle of Narayana Numbers arises from Dyck paths classified according to the number of peaks. Specifically, each entry $N(n,k)$ represents the number of Dyck paths of semilength n having exactly k peaks. This interpretation shows that the Narayana triangle provides a refined form of Catalan-type counting because it classifies the paths not only by size but also by their internal peak structure.

Proposition 1. For integers $n \geq 1$and $1 \leq k \leq n$, the Narayana number is given by

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

This formula confirms that the entries of the Narayana triangle can be expressed in terms of binomial coefficients. Hence, the triangle has a precise algebraic form while still maintaining its combinatorial meaning as a counting structure for Dyck paths with a specified number of peaks.

Structural Properties of the Narayana Triangle

The results further showed that the Narayana triangle possesses several important structural properties. Among these are its boundary values,

$$N(n,1) = N(n,n) = 1,$$

which indicate that for each row, the first and last entries are always equal to one. The triangle also exhibits symmetry, expressed as

$$N(n,k) = N(n,n-k+1).$$

This means that the entries in each row mirror one another, showing a balanced distribution of Dyck paths according to their number of peaks. These properties demonstrate that the Narayana triangle is not a random numerical arrangement but a highly organized combinatorial structure.

Relationship with Catalan Numbers

Another major result of the study is the relationship between Narayana numbers and Catalan numbers. The sum of the entries in the n-th row of the Narayana triangle is equal to the n-th Catalan number:

$$\sum_{k=1}^{n} N(n,k) = C_n.$$

Equivalently,

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

This result shows that the Catalan number gives the total number of Dyck paths of semilength n, while the Narayana numbers distribute this total according to the number of peaks. Therefore, the Narayana triangle serves as a refined version of Catalan enumeration.

Relationship with Pascal Triangle and Lah Numbers

The study also established that the Narayana triangle is related to other classical combinatorial arrays, particularly Pascal's triangle and unsigned Lah numbers. Its entries may be expressed through binomial coefficients, which connect it naturally with Pascal-type structures. Moreover, its relationship with the unsigned Lah numbers is expressed by

$$L(n,k) = (n-k+1)!\,N(n,k).$$

This formula shows that the Narayana numbers and unsigned Lah numbers are linked through a factorial transformation, further confirming that the Narayana triangle belongs to a broader network of combinatorial sequences.

Applications of Narayana Numbers

The findings also showed that Narayana numbers have applications beyond Dyck paths. They enumerate noncrossing set partitions of $[n]$ into $k$blocks, unlabeled ordered rooted trees with $n$nodes and k leaves, and

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permutations of $[n]$ avoiding the pattern 132 with exactly $k-1$descents. These applications demonstrate that the Narayana triangle provides a unifying counting framework for different combinatorial structures.

Overall, the results indicate that the Triangle of Narayana Numbers is both an algebraic and combinatorial structure. Its closed form, symmetry, boundary behavior, Catalan relationship, connection with Lah numbers, and applications in various counting problems confirm its significance in enumerative combinatorics.

CONCLUSION

This study concludes that the Triangle of Narayana Numbers is a significant combinatorial structure arising from Dyck paths classified according to the number of peaks. The results show that the Narayana numbers provide a refined form of Catalan counting by describing not only the total number of Dyck paths but also their internal peak distribution.

The study further confirms that the Narayana triangle possesses rich mathematical properties, including symmetry, boundary behavior, recurrence relations, parity patterns, generating functions, and structural identities. Its connections with Catalan numbers, Pascal's triangle, and unsigned Lah numbers demonstrate that it belongs to a broader family of important combinatorial arrays.

Moreover, the applications of Narayana numbers in noncrossing set partitions, unlabeled ordered rooted trees, and 132-avoiding permutations show that the triangle extends beyond Dyck paths and serves as a useful counting framework in enumerative combinatorics. Overall, the Triangle of Narayana Numbers may be regarded as both a refinement of Catalan enumeration and an independent mathematical object worthy of further investigation.

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