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Enumeration and Structural Analysis of Baxter Permutations

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ABSTRACT

This study investigated the enumeration and structural properties of Baxter permutations as a class of pattern-avoiding permutations within the symmetric group. It sought to define and verify the Baxter permutation sequence, determine the behavior of Baxter permutations under inverse, reversal, complement, reverse-complement, and standardization, and examine the enumeration of alternating and doubly alternating Baxter subclasses in relation to Catalan numbers. A theoretical approach combining descriptive, exploratory, and expository methods was employed. Definitions, examples, exact enumeration formulas, structural arguments, and proof-based analysis were synthesized from established combinatorial literature, while computed initial cases were compared with the Baxter number sequence recorded as OEIS A001181. The analysis confirmed that Baxter permutations are characterized by avoidance of the vincular patterns 2-41-3 and 3-14-2 and are enumerated by an exact binomial-coefficient summation. The class was shown to remain closed under inverse, reversal, complement, reverse-complement, and standardization. For alternating Baxter permutations, the source results give C_k^2 for even length 2k and C_kC_{k+1} for odd length 2k+1, while doubly alternating Baxter permutations of lengths 2k and 2k+1 are counted by C_k. These findings demonstrate the stability and enumerative richness of Baxter permutations and strengthen their connection with Catalan structures, bijective combinatorics, and broader discrete-mathematical constructions.

Keywords: *Baxter permutations, pattern avoidance, Baxter numbers, Catalan numbers, alternating permutations, enumerative combinatorics*

INTRODUCTION

Combinatorics studies discrete structures, arrangements, and counting principles, with permutations serving as one of its fundamental objects. Beyond the enumeration of all arrangements in the symmetric group S_n, pattern-avoiding permutation classes reveal how local restrictions produce global structural regularity. Baxter permutations occupy an important position in this area because they combine exact enumerative behavior with symmetry under several natural permutation operations.

Baxter permutations originated in Baxter's (1964) study of commuting functions and were subsequently developed as a distinct combinatorial class. They are characterized by avoidance of the vincular patterns 2-41-3 and 3-14-2. Chung et al. (1978) established their counting sequence, while later work clarified their pattern structure and connections with other discrete objects (Dulucq & Guibert, 1998). The resulting Baxter numbers are now catalogued as OEIS A001181 (The On-Line Encyclopedia of Integer Sequences [OEIS], n.d.), providing a standard reference for the initial terms of the sequence.

The importance of Baxter permutations extends beyond their raw enumeration. Bijections connect them with plane bipolar orientations and related planar structures (Bonichon et al., 2009; Felsner et al., 2011), while algebraic work places them within richer combinatorial frameworks such as Hopf-algebraic constructions

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(Giraudo, 2011). Their relationship with floorplans and tiling structures further illustrates how a constrained permutation class can encode geometric configurations (Ackerman et al., 2006; Korn, 2004).

The thesis addressed a more focused need: to bring enumeration, elementary structural operations, and Catalan-related subclasses into a unified treatment. Specifically, it defined and enumerated Baxter permutations; examined closure under inverse, reversal, complement, reverse-complement, and standardization; and developed the source study's enumerative formulas for alternating and doubly alternating Baxter permutations. This article condenses those mathematical results into a journal format while retaining the source definitions, formulas, examples, and conclusions.

Literature Review

Origins and Enumeration of Baxter Permutations

The modern study of Baxter permutations begins with Baxter (1964), whose work on fixed points of composites of commuting functions provided the setting from which the class emerged. Their subsequent interpretation as a pattern-avoiding family placed them within enumerative combinatorics, where counting restricted permutations is used to expose hidden structure. Chung et al. (1978) provided a foundational enumeration of the class and established the sequence now recognized as the Baxter numbers.

The counting perspective is reinforced by standard references on integer sequences and enumerative methods. Sloane (1973) and Sloane and Plouffe (1995) established systematic approaches for identifying and comparing integer sequences, while the OEIS (n.d.) records the Baxter numbers under A001181. More broadly, Stanley (1999) and Bóna (2015) situate restricted permutations and Catalan-type sequences within the wider theory of enumerative combinatorics, providing the conceptual background for interpreting exact formulas and initial terms.

Structural and Algebraic Characterizations

Dulucq and Guibert (1998) developed a systematic combinatorial treatment of Baxter permutations and helped formalize the restricted pattern structure that distinguishes the class from arbitrary elements of S_n. Pattern-avoidance language is especially useful because it allows membership in the class to be expressed through relative order conditions rather than through a generating procedure alone. Kitaev (2011) provides a broader framework for such generalized and vincular pattern restrictions in permutations and words.

Structural analysis also links Baxter permutations with algebraic combinatorics. Giraudo (2011) examined algebraic and combinatorial structures built from Baxter permutations, demonstrating that the class supports organization beyond numerical enumeration. In the source thesis, this structural viewpoint is made elementary and explicit through the analysis of inverse, reversal, complement, reverse-complement, and standardization, all of which preserve the defining avoidance conditions.

Catalan Subclasses and Bijective Connections

A major theme in the literature is the appearance of Catalan objects within restricted Baxter families. Guibert and Linusson (2000) proved that doubly alternating Baxter permutations are Catalan, establishing a direct connection between simultaneous alternation of a permutation and its inverse and one of the most pervasive sequences in combinatorics. This relationship provides the basis for the thesis treatment of alternating and doubly alternating subclasses.

Bijective methods explain why such formulas are structurally meaningful rather than merely numerical coincidences. Felsner et al. (2011) developed bijections for Baxter families and related objects, while Stanley (1999) discusses the ubiquity of Catalan structures such as Dyck paths and plane trees. The source study uses these Catalan correspondences to express alternating Baxter counts in product form and doubly alternating counts directly in terms of C_k.

Geometric and Computational Connections

Baxter permutations also encode geometric information. Bonichon et al. (2009) established a bijection with plane bipolar orientations, and Ackerman et al. (2006) connected permutation structures with floorplans and applications in discrete geometry. Korn (2004) examined related geometric and algebraic properties of polyomino tilings. Together, these works show that Baxter permutations can act as combinatorial descriptions of spatial arrangements, not simply as isolated sequences.

The computational side of enumeration is equally relevant because exact formulas must be evaluated, checked, and compared across increasing values of n. The source thesis used OEIS A001181 as a benchmark for initial Baxter numbers and emphasized formula-based enumeration over brute-force listing for larger cases. Exact summation techniques are part of the wider symbolic toolkit of enumerative combinatorics (Zeilberger, 1991), supporting the broader methodological context in which Baxter-number formulas are analyzed.

METHODS

Research Design

The study employed theoretical mathematical research using descriptive, exploratory, and expository methods. The descriptive component organized the definitions of permutations, pattern avoidance, Baxter permutations, Catalan numbers, and relevant permutation operations. The exploratory component examined structural behavior and subclass relationships, while the expository component presented theorem-based arguments, proofs, and illustrative cases. No experimental treatment or empirical participant data were involved.

Research Materials and Data Sources

The mathematical materials consisted of established books, journal articles, theses, and online sequence references cited in the source study. Particular attention was given to the foundational work of Baxter (1964), the enumeration of Chung et al. (1978), structural treatments by Dulucq and Guibert (1998), Catalan subclass results of Guibert and Linusson (2000), and the Baxter-number entry OEIS A001181. These sources supplied the definitions, previously established correspondences, and benchmark sequence values used in the analysis.

Data Gathering Procedure

Relevant definitions, theorems, formulas, and known Baxter-number values were gathered from the literature identified in the thesis. The study then organized these materials around three analytical tasks: defining and enumerating Baxter permutations, determining preservation under elementary permutation operations, and examining alternating and doubly alternating subclasses. Initial cases were worked explicitly to illustrate how the avoidance conditions and subclass definitions operate.

Data Analysis

Analysis proceeded through exact combinatorial enumeration and proof-based structural comparison. Baxter-number values were evaluated from the binomial-coefficient summation and compared with the sequence in OEIS A001181. Closure was examined by tracking how relative inequalities behave under inversion, reversal, complement, reverse-complement, and standardization. Subclass counts were interpreted through the source study's Catalan correspondences, with separate formulas for even and odd alternating Baxter permutations and a Catalan formula for doubly alternating Baxter permutations.

Ethical Consideration

Because the research was purely theoretical and used no human participants, animals, personal data, or experimental interventions, participant-based ethical procedures were not applicable. Scholarly integrity was addressed through attribution of definitions, established results, and prior bijections to the corresponding sources retained in the reference list.

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RESULTS AND DISCUSSION

Definition and Enumeration of Baxter Permutations

The study defined a Baxter permutation as a permutation π in S_n for which no indices $i < j < j + 1 < k$ satisfy either of two forbidden relative-order conditions. These conditions are the vincular patterns 2-41-3 and 3-14-2 and distinguish the Baxter subset B_n from the complete symmetric group S_n.

B1: $\pi_{j+1} < \pi_i < \pi_k < \pi_j$, or

B2: $\pi_j < \pi_k < \pi_i < \pi_{j+1}$.

For $n \leq 3$, no four-position configuration can occur, so every permutation is Baxter. At $n = 4$, the distinction becomes nontrivial: among the 24 permutations in S_4, the source study identified exactly two non-Baxter permutations, (2, 4, 1, 3) and (3, 1, 4, 2), leaving |B_4| = 22. This agrees with the Baxter sequence reported by Chung et al. (1978) and OEIS A001181.

Table 1. *Initial Baxter numbers compared with the symmetric group*

n	$\lvert S_n \rvert = n!$	$\lvert B_n \rvert$
1	1	1
2	2	2
3	6	6
4	24	22
5	120	92
6	720	422
7	5040	2074
8	40320	10754

The exact enumeration used in the thesis is expressed through a finite sum of products of binomial coefficients. This formulation reproduces the initial sequence 1, 2, 6, 22, 92, 422, 2074, 10754, … and avoids direct listing of all n! permutations as n grows. The formula retained from the thesis is:

$$|\mathfrak{B}_n| = \sum_{k=1}^{n} \frac{\binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}}{\binom{n+1}{1}\binom{n+1}{2}}.$$

The numerator selects adjacent layers around the summation index, whereas the denominator provides the normalization stated in the source theorem. Verification of the first cases and the value B_5 = 92 matched OEIS A001181, supporting the consistency of the enumeration with established Baxter numbers (Chung et al., 1978; OEIS, n.d.).

Structural Closure Under Permutation Operations

The central structural result of the study is that the Baxter class is preserved under several natural operations. Because the defining restrictions depend on relative order, operations that systematically transform positions or values preserve the avoidance structure in the ways demonstrated by the source proofs. Inversion exchanges the roles of positions and values; reversal reflects positions; complement reflects values; and reverse-complement composes the latter two transformations.

Table 2. Structural properties established for Baxter permutations

Operation	Source result	Structural basis
Inverse	Closed	Relative order is preserved under exchanging positions and values.
Reversal	Closed	Reversing positions maps forbidden configurations symmetrically.
Complement	Closed	Value reflection transforms inequalities without creating a forbidden Baxter pattern.

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Reverse-complement	Closed	Closure follows from successive reversal and complement.
Standardization	Preserved	Standardization retains all relative comparisons among distinct entries.

Standardization is particularly important because it shows that the Baxter property depends on order type rather than on the particular numerical labels assigned to entries. This aligns the study with the broader pattern-avoidance viewpoint developed by Dulucq and Guibert (1998) and discussed within generalized permutation-pattern theory (Kitaev, 2011). The resulting stability also helps explain why Baxter permutations admit richer algebraic organization (Giraudo, 2011).

Alternating Baxter Permutations

The study next restricted the Baxter class by imposing the alternating pattern $\pi_1 < \pi_2 > \pi_3 < \pi_4 > \cdots$. Explicit inspection gave one alternating Baxter permutation at n = 2, two at n = 3, and four at n = 4. The thesis then expressed the counts in terms of Catalan numbers, separating even and odd lengths.

For even length n = 2k, the source theorem gives:

$$\left|\mathfrak{B}_{2k}^{alt}\right| = C_k^2$$

$$C_k = \frac{1}{k+1}\binom{2k}{k} \text{ is the } k-th \text{ Catalan number.}$$

For odd length n = 2k + 1, the corresponding formula is:

$$\left|\mathfrak{B}_{2k+1}^{alt}\right| = C_k \cdot C_{k+1}$$

Thus, the parity of the permutation length determines whether the count is the square of one Catalan number or the product of two consecutive Catalan numbers. The thesis interprets this through paired Dyck-path or twin-tree structures, consistent with the Catalan connections documented in the Baxter literature (Guibert & Linusson, 2000; Felsner et al., 2011).

Doubly Alternating Baxter Permutations

A doubly alternating Baxter permutation must be alternating and must have an inverse that is also alternating. This further restriction yields a cleaner Catalan enumeration. In the source example for n = 3, only (1, 3, 2) satisfied both the alternating condition and the corresponding condition on its inverse.

For every k ≥ 0, the source result states that the doubly alternating counts at consecutive even and odd lengths are equal to the k-th Catalan number:

$$\left|\mathfrak{B}_{2k}^{doub}\right| = \left|\mathfrak{B}_{2k+1}^{doub}\right| = C_k$$

This result places the doubly alternating subclass directly within a Catalan family. It is consistent with Guibert and Linusson (2000), who established the Catalan enumeration of doubly alternating Baxter permutations, and with the broader use of Catalan objects such as Dyck paths and binary trees in enumerative combinatorics (Stanley, 1999).

Table 3. *Enumerative formulas for Baxter subclasses retained from the thesis*

Subclass	Length	Count
Alternating Baxter	n = 2k	C_k^2
Alternating Baxter	n = 2k + 1	C_kC_{k+1}
Doubly alternating Baxter	n = 2k or 2k + 1	C_k

Taken together, the results show three complementary layers of structure. First, the Baxter numbers provide an exact enumeration of a restricted permutation class. Second, closure under elementary operations demonstrates structural stability. Third, alternating restrictions expose a direct Catalan substructure. These layers explain why Baxter permutations recur in bijective, geometric, and algebraic settings such as plane bipolar orientations, floorplans, and related Baxter families (Ackerman et al., 2006; Bonichon et al., 2009; Felsner et al., 2011).

CONCLUSION

Baxter permutations form a well-defined and highly structured subset of the symmetric group characterized by avoidance of the vincular patterns 2-41-3 and 3-14-2. The source enumeration formula reproduces the Baxter sequence recorded as OEIS A001181, while the structural analysis shows that the class is preserved under inverse, reversal, complement, reverse-complement, and standardization. These results demonstrate that the defining property is stable under fundamental transformations of permutation order.

The subclass analysis further establishes a strong connection with Catalan-numbered families. Alternating Baxter permutations are counted by C_k^2 for length 2k and by C_kC_{k+1} for length 2k + 1, whereas doubly alternating Baxter permutations of lengths 2k and 2k + 1 are counted by C_k. Collectively, these findings position Baxter permutations as an important bridge among pattern avoidance, exact enumeration, structural symmetry, and Catalan combinatorics.

Recommendation

Future research may extend the thesis by developing more efficient generation algorithms for larger Baxter classes and by testing the source enumeration through higher-order computational cases. Additional subclasses, including Baxter involutions, derangements, and further pattern-restricted families, may be examined to determine whether they produce other classical integer sequences or admit new bijections.

The closure results also support deeper study of algebraic structures associated with Baxter permutations, including group-theoretic and Hopf-algebraic constructions. Finally, the established connections with plane orientations, tilings, and floorplans may be explored further in discrete geometry and computational applications, provided that such extensions remain grounded in rigorously established combinatorial correspondences.

References

Ackerman, E., Barequet, G., & Pinter, R. Y. (2006). A bijection between permutations and floorplans, and its applications. *Discrete Applied Mathematics, 154*(12), 1674–1684. https://doi.org/10.1016/j.dam.2006.03.018

Baxter, G. (1964). On fixed points of the composite of commuting functions. *Proceedings of the American Mathematical Society, 15*(6), 851–855. https://doi.org/10.2307/2034894

Bóna, M. (2015). *Handbook of enumerative combinatorics*. CRC Press. https://doi.org/10.1201/B18255

Bonichon, N., Bousquet-Mélou, M., & Fusy, É. (2009). Baxter permutations and plane bipolar orientations. *Séminaire Lotharingien de Combinatoire, 61A*, Article B61Ah. https://arxiv.org/abs/0805.4180

Chung, F. R. K., Graham, R. L., Hoggatt, V. E., Jr., & Kleiman, M. (1978). The number of Baxter permutations. *Journal of Combinatorial Theory, Series A, 24*(3), 382–394. https://doi.org/10.1016/0097-3165(78)90068-7

Dulucq, S., & Guibert, O. (1998). Baxter permutations. *Discrete Mathematics, 180*(1–3), 143–156. https://doi.org/10.1016/S0012-365X(97)00112-X

Felsner, S., Fusy, É., Noy, M., & Orden, D. (2011). Bijections for Baxter families and related objects. *Journal of Combinatorial Theory, Series A, 118*(3), 993–1020.

Giraudo, S. (2011). Algebraic and combinatorial structures on Baxter permutations. *Proceedings of the 23rd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2011)*, 387–398. https://arxiv.org/abs/1011.4288

Guibert, O., & Linusson, S. (2000). Doubly alternating Baxter permutations are Catalan. *Discrete Mathematics, 217*(1–3), 157–166. https://doi.org/10.1016/S0012-365X(99)00261-7

Kitaev, S. (2011). *Patterns in permutations and words*. Springer-Verlag.

Korn, M. (2004). *Geometric and algebraic properties of polyomino tilings* (Doctoral dissertation, Massachusetts Institute of Technology).

Sloane, N. J. A. (1973). *A handbook of integer sequences*. Academic Press.

Sloane, N. J. A., & Plouffe, S. (1995). *The encyclopedia of integer sequences*. Academic Press.

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of Education, Research, and Innovation Perspectives
Volume 2 Issue 9 (June 2026)

Stanley, R. P. (1999). *Enumerative combinatorics* (Vol. 2). Cambridge University Press.
The On-Line Encyclopedia of Integer Sequences. (n.d.). *A001181: Baxter numbers*. https://oeis.org/A001181
Zeilberger, D. (1991). The method of creative telescoping. *Journal of Symbolic Computation, 11*(3), 195–204.